REFLECTED STABLE SUBORDINATORS FOR FRACTIONAL CAUCHY PROBLEMS

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Abstract. In a fractional Cauchy problem, the first time derivative is replaced by a Caputo fractional derivative of order less than one. If the original Cauchy problem governs a Markov process, a non-Markovian time change yields a stochastic solution to the fractional Cauchy problem, using the first passage time of a stable subordinator. This paper proves that a spectrally negative stable process reflected at its infimum has the same one dimensional distributions as the inverse stable subordinator. Therefore, this Markov process can also be used as a time change, to produce stochastic solutions to fractional Cauchy problems. The proof uses an extension of the D. André reflection principle. The forward equation of the reflected stable process is established, including the appropriate fractional boundary condition, and its transition densities are explicitly computed.

1. Introduction

A classical paper of Einstein [13] outlines the link between random walks, Brownian motion, and the diffusion equation $\partial_t u(x,t) = \partial_x^2 u(x,t)$. Sokolov and Klafter [33] review modern extensions of the theory to encompass random walks with heavy tails, jump processes, and fractional diffusion. This connection between probability and differential equations has profound consequences for mathematics [1, 7, 8, 25], and for its applications in science and engineering [16, 27, 28, 31], including a probabilistic method for solving fractional differential equations, by exploiting the associated Markov process [37]. More details on fractional calculus, and its connection to probability theory, may be found in the recent book of Meerschaert and Sikorskii [26].

The time-fractional diffusion equation $\partial_t^\beta u(x,t) = \partial_x^2 u(x,t)$ with $0 < \beta < 1$ governs the long-time limit of a random walk with power law waiting times $\mathbb{P}(W > t) \sim Ct^{-\beta}$ between particle jumps [22, Theorem 4.2], see also [26, Sec. 4.5]. The limit process is a Brownian motion, time-changed by the inverse or hitting time E_t of a β-stable subordinator. This time change process is not Markovian, since it remains constant over time intervals whose distribution is not exponential.

The time-fractional diffusion equation is one example of a fractional Cauchy problem. More generally, for any strongly continuous Markov semigroup $T_t f(x) = \mathbb{E}^x[f(X_t)]$ with generator L, it follows from [3, Theorem 3.1] along with [22, Corollary 3.1] that

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$p(x,t) = \mathbb{E}^x[f(X_{E_t})]$ solves the fractional Cauchy problem

$$\partial_t^\beta p(x,t) = Lp(x,t); \quad p(x,0) = f(x) \tag{1.1}$$

of order $0 < \beta < 1$, for any f in the domain of the generator L. The Caputo fractional derivative in (1.1) is defined by

$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^\infty f'(t-r) r^{-\beta} dr. \tag{1.2}$$

The stochastic process X_{E_t} provides the basis for particle tracking solutions to the time-fractional Cauchy problem, in which the underlying process is simulated, and the resulting histogram for many sample paths approximates the solution [21, 38].

The goal of this paper is to provide an alternative method for particle tracking solutions of fractional Cauchy problems, by establishing an equivalent solution method in terms of a Markovian time change. This solves an open problem in [5, Remark 5.2]. We will show that a certain Markov process Z_t has the same one dimensional distributions as the inverse stable subordinator E_t, and therefore, one can also compute solutions to fractional Cauchy problems by the probabilistic formula $p(x,t) = \mathbb{E}^x[f(X_{Z_t})]$. The proof uses a variation of the D. André reflection principle, which may also be useful in other contexts. The Markov process Z_t is a stable Lévy process with index $\alpha = 1/\beta$ and no positive jumps, reflected at the origin. At the end of this paper, we establish the forward (Fokker-Planck) equation for this Markov process, which involves a fractional boundary condition, and we explicitly compute the transition densities (see Figures 1 and 2).

Notation. Most of our notation is standard or self-explicatory. Since our paper addresses both probabilists and analysts, some words on function spaces and convergence seem to be in order. We write $C_\infty[0,\infty)$ for the continuous functions which vanish at infinity, i.e., $\lim_{x\to\infty} u(x) = 0$, and this space is always equipped with uniform norm $\|u\| = \sup_{x\in[0,\infty)} |u(x)|$. Its topological dual is the space of (signed) Radon measures $\mathcal{M}_b[0,\infty)$, and by $\mathcal{M}_{ac}[0,\infty)$ we mean the absolutely continuous (with respect to Lebesgue measure) elements in $\mathcal{M}_b[0,\infty)$. On $\mathcal{M}_b[0,\infty)$ we use vague convergence which is, topologically, the weak-* convergence; i.e., $\mu_n \to \mu$ vaguely if, and only if, $\int u\, d\mu_n \to \int u\, d\mu$ for all $u \in C_\infty[0,\infty)$. The subscripts c, b, ac, ∞ stand for 'compact support', 'bounded', 'absolutely continuous' and 'vanishing at infinity'.

Fractional integrals and derivatives in the Riemann-Liouville sense are denoted by I^α and D^α, cf. (4.2)–(4.5) while Caputo derivatives are written as ∂^α, cf. (4.11).

2. Reflection principle

In this section, we note a useful extension of the D. André reflection principle for Brownian motion. Since this extension is not completely standard, we include a simple proof.

Theorem 2.1 (reflection principle)**.** *Suppose that Y_t is a Lévy process started at the origin, with no positive jumps, and let $S_t = \sup\{Y_u : 0 \leqslant u \leqslant t\}$. Assume that*

$\mathbb{P}(Y_t > 0) = \mathbb{P}(Y_1 \geqslant 0)$, *for all* $t > 0$. *Then*

$$\mathbb{P}(S_t \geqslant x) = \mathbb{P}(Y_t \geqslant x \,|\, Y_t \geqslant 0) = \frac{\mathbb{P}(Y_t \geqslant x)}{\mathbb{P}(Y_t \geqslant 0)} \tag{2.1}$$

for all $t, x > 0$.

Proof. Let $\tau_x := \inf\{u > 0 \;:\; Y_u > x\}$ denote the first passage time process. Since $(Y_t)_{t\geqslant 0}$ has stationary independent increments, it follows that $(Y_{t+\tau_x} - Y_{\tau_x})_{t\geqslant 0}$ is a Lévy process, which is independent of the σ-algebra generated by $(Y_t)_{t\leqslant \tau_x}$, and it has the same finite dimensional distributions as $(Y_t)_{t\geqslant 0}$. Consequently

$$\begin{aligned} \mathbb{P}\left(\tau_x \leqslant t, Y_t < Y_{\tau_x}\right) &= \mathbb{P}\left(\tau_x \leqslant t, Y_t - Y_{\tau_x} < 0\right) \\ &= \mathbb{P}\left(\tau_x \leqslant t\right) \mathbb{P}\left(Y_1 < 0\right) \\ &= \frac{\mathbb{P}\left(Y_1 < 0\right)}{\mathbb{P}\left(Y_1 \geqslant 0\right)} \mathbb{P}\left(\tau_x \leqslant t\right) \mathbb{P}\left(Y_1 \geqslant 0\right) \\ &= \frac{\mathbb{P}\left(Y_1 < 0\right)}{\mathbb{P}\left(Y_1 \geqslant 0\right)} \mathbb{P}\left(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x}\right). \end{aligned} \tag{2.2}$$

Observe that we have for any càdlàg process

$$\{\tau_x < t\} \subset \{S_t > x\} \subset \{\tau_x \leqslant t\} \subset \{S_t \geqslant x\}$$

for all t and $x > 0$. Therefore,

$$\begin{aligned} \mathbb{P}(S_t > x) \leqslant \mathbb{P}(\tau_x \leqslant t) &= \mathbb{P}(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x}) + \mathbb{P}(\tau_x \leqslant t, Y_t < Y_{\tau_x}) \\ &= \left(1 + \frac{\mathbb{P}\left(Y_1 < 0\right)}{\mathbb{P}\left(Y_1 \geqslant 0\right)}\right) \mathbb{P}(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}\left(Y_1 \geqslant 0\right)} \mathbb{P}(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}\left(Y_t \geqslant 0\right)} \mathbb{P}(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}\left(Y_t \geqslant 0\right)} \mathbb{P}(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x}, Y_t \geqslant 0) \\ &= \mathbb{P}(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x} \,|\, Y_t \geqslant 0). \end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{P}(S_t > x) \geqslant \mathbb{P}(\tau_x < t) &= \mathbb{P}(\tau_x < t, Y_t \geqslant Y_{\tau_x}) + \mathbb{P}(\tau_x < t, Y_t < Y_{\tau_x}) \\ &= \left(1 + \frac{\mathbb{P}\left(Y_1 < 0\right)}{\mathbb{P}\left(Y_1 \geqslant 0\right)}\right) \mathbb{P}(\tau_x < t, Y_t \geqslant Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}\left(Y_1 \geqslant 0\right)} \mathbb{P}(\tau_x < t, Y_t \geqslant Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}\left(Y_t \geqslant 0\right)} \mathbb{P}(\tau_x < t, Y_t \geqslant Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}\left(Y_t \geqslant 0\right)} \mathbb{P}(\tau_x < t, Y_t \geqslant Y_{\tau_x}, Y_t \geqslant 0) \\ &= \mathbb{P}(\tau_x < t, Y_t \geqslant Y_{\tau_x} \,|\, Y_t \geqslant 0) \\ &\geqslant \mathbb{P}(\tau_x < t, Y_t > Y_{\tau_x} \,|\, Y_t \geqslant 0).\end{aligned}$$

Therefore,

$$\mathbb{P}(\tau_x < t, Y_t > Y_{\tau_x} \,|\, Y_t \geqslant 0) \leqslant \mathbb{P}(S_t > x) \leqslant \mathbb{P}(\tau_x \leqslant t, Y_t \geqslant Y_{\tau_x} \,|\, Y_t \geqslant 0).$$

Since Y_t has no upward jumps, $Y_{\tau_x} = x$ and $\{\tau_x < t\} \cap \{Y_t > x\} = \{Y_t > x\}$. Therefore,

$$\mathbb{P}(Y_t > x \,|\, Y_t \geqslant 0) \leqslant \mathbb{P}(S_t > x) \leqslant \mathbb{P}(Y_t \geqslant x \,|\, Y_t \geqslant 0).$$

We can now use standard approximation techniques:

$$\{X \geqslant x\} = \bigcap_n \{X > x - 1/n\} = \bigcap_n \{X \geqslant x - 1/n\}$$

and

$$\{X > x\} = \bigcup_n \{X > x + 1/n\}$$

to get

$$\begin{aligned}\mathbb{P}(Y_t \geqslant x \,|\, Y_t \geqslant 0) &= \lim_{n\to\infty} \mathbb{P}(Y_t > x - 1/n \,|\, Y_t \geqslant 0) \\ &\leqslant \lim_{n\to\infty} \mathbb{P}(S_t > x - 1/n) = \mathbb{P}(S_t \geqslant x)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(S_t \geqslant x) = \lim_{n\to\infty} \mathbb{P}(S_t > x - 1/n) &\leqslant \lim_{n\to\infty} \mathbb{P}(Y_t \geqslant x - 1/n \,|\, Y_t \geqslant 0) \\ &= \mathbb{P}(Y_t \geqslant x \,|\, Y_t \geqslant 0)\end{aligned}$$

which proves $\mathbb{P}(S_t \geqslant x) = \mathbb{P}(Y_t \geqslant x \,|\, Y_t \geqslant 0)$. □

Remark 2.2. If $(Y_t)_{t\geqslant 0}$ is a Brownian motion, then (2.1) becomes the classical reflection principle: $\mathbb{P}(Y_t \geqslant 0) = 1/2$, so that (2.1) is equivalent to $\mathbb{P}(S_t \geqslant x) = 2\mathbb{P}(Y_t \geqslant x)$.

The proof of Theorem 2.1 relies essentially on local symmetry and the strong Markov property. Let Y_t be a strong Markov process with càdlàg paths and transition function $p_t(z, dy) = \mathbb{P}^z(Y_t \in dy)$, and write $\tau_x^z = \inf\{u > 0 : Y_u - z > x\}$ for the first passage time above the level $x + z$ for the process Y_t started at z; observe that, in general,

$Y_{\tau_x^z} \geqslant x+z$. We can use the strong Markov property in (2.2) to get for any starting point z

$$\begin{aligned}\mathbb{P}^z\left(\tau_x^z \leqslant t, Y_t < Y_{\tau_x^z}\right) &= \mathbb{P}^z\left(\tau_x^z \leqslant t, Y_t - Y_{\tau_x^z} < 0\right) \\ &= \int_{\{\tau_x^z \leqslant t\}} \mathbb{P}^{Y_{\tau_x^z}(\omega)}\left(Y_{t-\tau_x^z(\omega)} - Y_0 < 0\right) \mathbb{P}^z(d\omega).\end{aligned}$$

If we assume, in addition, some local 'symmetry', i.e., that for some constant $c \in (0,\infty)$ we have

$$\frac{\mathbb{P}^z(Y_t - z < 0)}{\mathbb{P}^z(Y_t - z \geqslant 0)} = c \quad \text{for all } t>0,\ z \in \mathbb{R}, \tag{2.3}$$

then we get $\mathbb{P}^{Y_{\tau_x^z}(\omega)}\left(Y_{t-\tau_x^z(\omega)} - Y_0 < 0\right) = c\,\mathbb{P}^{Y_{\tau_x^z}(\omega)}\left(Y_{t-\tau_x^z(\omega)} - Y_0 \geqslant 0\right)$ and, with a similar argument,

$$\mathbb{P}^z\left(\tau_x^z \leqslant t, Y_t < Y_{\tau_x^z}\right) = c\,\mathbb{P}^z\left(\tau_x^z \leqslant t, Y_t \geqslant Y_{\tau_x^z}\right).$$

This means that we can follow the lines of the proof of Theorem 2.1 to derive the following general result.

Theorem 2.3 (Markov reflection principle)**.** *Suppose* $(Y_t, \mathbb{P}^z)$ *is a strong Markov process satisfying the local symmetry condition* (2.3)*. Set* $S_t = \sup\{Y_u - Y_0 : 0 \leqslant u \leqslant t\}$*. Then we have for all* $t, x > 0$ *and* $z \in \mathbb{R}$

$$\begin{aligned}\mathbb{P}^z(S_t > x) &\leqslant \mathbb{P}^z(S_t \geqslant x, Y_t \geqslant Y_{\tau_x^z} \,|\, Y_t \geqslant z) \\ &\leqslant \mathbb{P}^z(Y_t - z \geqslant x \,|\, Y_t \geqslant z).\end{aligned} \tag{2.4}$$

If Y_t *has only non-positive jumps, then* $Y_{\tau_x^z} = x+z$ *a.s., and we get for all* $t, x > 0$ *and* $z \in \mathbb{R}$

$$\mathbb{P}^z(S_t \geqslant x) = \mathbb{P}^z(Y_t - z \geqslant x \,|\, Y_t \geqslant z). \tag{2.5}$$

3. An equivalent Markov time change

Suppose that D_t is a standard stable subordinator, a Lévy process with $\mathbb{E}[e^{-sD_t}] = e^{-ts^\beta}$ for all $t \geqslant 0$, and define its inverse (hitting time, first passage time) process by

$$E_t = \inf\{r > 0 : D_r > t\} \tag{3.1}$$

for all $t \geqslant 0$. In this section, we prove that a certain Markov process Z_t has the same one dimensional distributions as the inverse stable subordinator (3.1) with index $\beta \in [1/2, 1)$. Then we use the Markov process Z_t as a time change, to develop efficient probabilistic solutions to fractional Cauchy problems.

To construct this Markov process, first consider a stable Lévy process Y_t with characteristic function

$$\mathbb{E}[e^{ikY_t}] = e^{t(ik)^\alpha} \tag{3.2}$$

where $\alpha = 1/\beta$. If $\beta = 1/2$, then $\alpha = 2$, and Y_t is a Brownian motion with variance $2t$. Now define

$$Z_t = Y_t - \inf\{Y_s : 0 \leqslant s \leqslant t\}. \tag{3.3}$$

The reflected stable process (3.3) is also the recurrent extension of the process Y_t killed at zero, which instantaneously and continuously leaves zero, see Patie and Simon [29]. Let $C_\infty(\mathbb{R})$ denote the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that tend to zero as $|x| \to \infty$, with the supremum norm. We say that a time-homogeneous Markov process X_t is a *Feller process* if the semigroup $T_t f(x) = \mathbb{E}[f(X_{t+s})|X_s = x]$ satisfies $T_t f \in C_\infty(\mathbb{R})$ and $T_t f \to f$ as $t \to 0$ in the Banach space (supremum) norm, for all $f \in C_\infty(\mathbb{R})$.

Lemma 3.1. *Let E_t be the inverse stable subordinator* (3.1)*, where D_t is a standard β-stable subordinator with $\mathbb{E}[e^{-sD_t}] = e^{-ts^\beta}$ and $\beta \in [1/2, 1)$. Define Z_t by* (3.3)*, where Y_t is a stable Lévy process with characteristic function* (3.2)*, with index $\alpha = 1/\beta$. Then Z_t is a Feller process, with the same (one-dimensional) marginal distributions (given $Z_0 = 0$) as E_t, for any fixed $t \geqslant 0$.*

Proof. Define the running infimum $I_t = \inf\{Y_s : 0 \leqslant s \leqslant t\}$ and the running supremum $S_t = \sup\{Y_s : 0 \leqslant s \leqslant t\}$. Let $\hat{Y}_t = -Y_t$ denote the dual process, and let $\hat{I}_t$ and $\hat{S}_t$ denote the running infimum and supremum of $\hat{Y}_t$, respectively. Since $\hat{Y}_t$ is also a Lévy process, it follows from [10, Section VI.1, Proposition 1] that $\hat{S}_t - \hat{Y}_t$ is a Feller process, and since $\hat{S}_t = -I_t$, it follows that $\hat{S}_t - \hat{Y}_t = -I_t + Y_t = Z_t$. Hence Z_t is a Feller process. Since the standard stable subordinator D_t has characteristic function $\mathbb{E}[e^{ikD_t}] = \exp(-t(-ik)^\gamma)$, an application of the Zolotarev duality for stable densities [5, Theorem 4.1] implies that

$$\mathbb{P}(E_t > x) = \mathbb{P}(Y_t > x | Y_t \geqslant 0) \quad \text{for all } t > 0 \text{ and all } x > 0. \tag{3.4}$$

The stable Lévy process Y_t has no positive jumps, and since Y_t is self-similar (e.g., see [26, p. 105]), it follows that $\mathbb{P}(Y_t > 0) = \mathbb{P}(t^{1/\alpha} Y_1 > 0) = \mathbb{P}(Y_1 > 0)$ for all $t > 0$. Since every stable process has a Lebesgue density (e.g., see [26, p. 107]), we also have $\mathbb{P}(Y_t = 0) = 0$ for all $t > 0$. Then it follows from Theorem 2.1 that

$$\mathbb{P}(Y_t > x | Y_t \geqslant 0) = \mathbb{P}(S_t \geqslant x) \quad \text{for all } t > 0 \text{ and all } x > 0. \tag{3.5}$$

Finally, a fluctuation identity [10, Section VI.1, Prop. 3] implies that

$$\mathbb{P}(S_t \geqslant x) = \mathbb{P}(Z_t \geqslant x) \quad \text{for all } t > 0 \text{ and all } x > 0. \tag{3.6}$$

Then the theorem follows by combining (3.4), (3.5), and (3.6). □

Remark 3.2. It follows from the Zolotarev duality formula for stable laws [20, 39] that in fact $\mathbb{P}(Y_t \geqslant 0) = 1/\alpha$, see [5, Theorem 4.1].

The next result shows that fractional Cauchy problems can be solved by a probabilistic method, using a Markovian time change.

Theorem 3.3. *For any $\beta \in [1/2, 1)$, let Z_t be given by* (3.3)*, where Y_t is an α-stable Lévy process with characteristic function* (3.2) *for $\alpha = 1/\beta$. If X_t is an independent Markov process such that $T_t f(x) = \mathbb{E}^x[f(X_t)]$ forms a uniformly bounded, strongly continuous semigroup on some Banach space $\mathbb{B}$ of real valued functions, with generator L, then $p(x,t) = \mathbb{E}^x[f(X_{Z_t})]$ solves the fractional Cauchy problem* (1.1) *for any $f \in$ D(L), the domain of the generator.*

Proof. A general result [3, Theorem 3.1] shows that, if L is the generator of a uniformly bounded, strongly continuous semigroup on some Banach space $\mathbb{B}$, and if $u(\cdot, t) \in \mathbb{B}$ solves the Cauchy problem $\partial_t u(x,t) = Lu(x,t)$; $u(x,0) = f(x)$ on that space for some $f \in \mathrm{Dom}(L)$, then the solution to the associated fractional Cauchy problem (1.1) on that Banach space is given by

$$p(x,t) = \int_0^\infty u(x,r) h(r,t)\, dr \tag{3.7}$$

where

$$h(r,t) = \frac{t}{\beta} r^{-1-1/\beta}\, g_\beta(t r^{-1/\beta}) \tag{3.8}$$

and $g_\beta(t)$ is the function with Laplace transform

$$\mathcal{L}g(s) = \int_0^\infty e^{-st} g_\beta(t)\, dt = e^{-s^\beta}$$

for some $0 < \beta < 1$. Of course $g_\beta(t)$ is the probability density function of a standard β-stable subordinator. Let D_t be the associated Lévy process with $\mathbb{E}[e^{-sD_t}] = e^{-ts^\beta}$ and apply [22, Corollary 3.1] to see that (3.8) is also the probability density of the hitting time (3.1). Since $T_t f(x) = \mathbb{E}^x[f(X_t)]$ is a uniformly bounded, strongly continuous semigroup, it follows that $p(x,t) = \mathbb{E}^x[f(X_{E_t})]$ solves the fractional Cauchy problem (1.1). Lemma 3.1 implies that the random variables E_t and Z_t have the same distribution for any $t > 0$, and we also have $Z_0 = E_0 = 0$ almost surely. Then it follows that $\mathbb{E}^x[f(X_{Z_t})] = \mathbb{E}^x[f(X_{E_t})]$, which completes the proof. $\square$

Remark 3.4. Theorem 3.3 confirms a conjecture in the paper [5, Remark 5.2]. There we set $Z_t = Y_{\sigma(t)}$ where $\sigma(t) = \inf\{u > 0 : H_u > t\}$ and $H_u = \int_0^u \mathbb{1}_{Y_s > 0}\, ds$. In essence, the negative excursions are cut away, and the positive excursions are joined together without any gaps in time. Since Y_t has no positive jumps, any up-crossing at the origin is a renewal point, so this process has the same distribution as (3.3).

Remark 3.5. Since the stable Lévy process Y_t with index $\alpha \in (1,2]$ and characteristic function (3.2) has no positive jumps, the first passage time $D_t := \inf\{r > 0; Y_r > t\}$ is a stable subordinator with $\mathbb{E}[e^{-sD_t}] = \exp(-ts^{1/\alpha})$, and the supremum process $E_t = \sup\{Y_r : 0 \leqslant r \leqslant t\}$ is the inverse stable process (3.1) of D_t, see Bingham [12]. This allows us to compare the processes Z_t and E_t in terms of their sample paths. It can also be used to give an alternative proof of Lemma 3.1: Since $S_t = E_t$, we certainly have $\mathbb{P}(E_t \geqslant x) = \mathbb{P}(S_t \geqslant x)$, and then the result follows by the fluctuation identity (3.6). For the case $\alpha = 2$, Z_t is the reflected Brownian motion, and E_t is the supremum of that same Brownian motion.

4. The reflected stable process

The equivalence of one dimensional distributions established in Lemma 3.1 also has some interesting implications regarding the reflected stable process Z_t defined in (3.3).

Proposition 4.1. *Let Y_t be a stable Lévy process with characteristic function* (3.2) *and index $\alpha \in (1,2)$. Then the reflected process Z_t in* (3.3) *has the right-continuous probability density*

$$p(x,t) = \begin{cases} t\beta^{-1}x^{-1-1/\beta}g_\beta(tx^{-1/\beta}) & x > 0 \\ t^{-\beta}/\Gamma(1-\beta) & x = 0 \\ 0 & x < 0 \end{cases} \tag{4.1}$$

for any $t > 0$, where $\beta = 1/\alpha$ and $g_\beta(t)$ is the probability density function with Laplace transform $\mathcal{L}g(s) = \exp(-s^{1/\alpha})$.

Proof. Lemma 3.1 shows that the probability distribution of Z_t is also the probability distribution of the random variable E_t, the hitting time (3.1) of a stable subordinator D_t such that $\mathbb{E}[e^{-sD_t}] = \exp(-ts^\beta)$, and $\beta = 1/\alpha$. Then [22, Corollary 3.1] implies that E_t has a Lebesgue density $h(x,t) = t\beta^{-1}x^{-1-1/\beta}g_\beta(tx^{-1/\beta})$ for $x > 0$. Since D_t is right-continuous, it follows from (3.1) that $E_t > 0$ for $t > 0$, so that $p(x,t) = 0$ for $x < 0$ and $t > 0$. Finally, since $g_\beta(x) \sim \beta x^{-\beta-1}/\Gamma(1-\beta)$ as $x \to \infty$, it is easy to check that $p(x,t) \to t^{-\beta}/\Gamma(1-\beta)$ as $x \to 0+$ for any $t > 0$. □

The next result shows that the density (4.1) of Z_t started at $Z_0 = 0$ solves a fractional boundary value problem. Given a real number $\alpha > 0$ which is not an integer, define the positive Riemann-Liouville fractional integral

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty f(x-y)y^{\alpha-1}\,dy = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x f(y)(x-y)^{\alpha-1}\,dy, \tag{4.2}$$

the negative Riemann-Liouville fractional integral

$$I_{-x}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty f(x+y)y^{\alpha-1}\,dy = \frac{1}{\Gamma(\alpha)}\int_x^\infty f(y)(y-x)^{\alpha-1}\,dy, \tag{4.3}$$

the positive Riemann-Liouville fractional derivative

$$D_x^\alpha f(x) := \frac{d^n}{dx^n}I_x^{n-\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^x f(y)(x-y)^{n-\alpha-1}\,dy, \tag{4.4}$$

and the negative Riemann-Liouville fractional derivative

$$D_{-x}^\alpha f(x) := \frac{d^n}{d(-x)^n}I_{-x}^{n-\alpha}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^\infty f(y)(y-x)^{n-\alpha-1}\,dy, \tag{4.5}$$

where $n-1 < \alpha < n$. If $\alpha \in (1,2)$, then $n = 2$.

Theorem 4.2. *Let Y_t be a stable Lévy process with characteristic function* (3.2) *and index $1 < \alpha < 2$, and let Z_t be the reflected process* (3.3)*. Then the probability density* (4.1) *of Z_t solves the fractional boundary value problem:*

$$\begin{aligned} \partial_t p(x,t) &= D_{-x}^\alpha p(x,t) \quad \text{for } t > 0 \text{ and } x > 0; \\ 0 &= D_{-x}^{\alpha-1}p(x,t) \quad \text{for } t > 0 \text{ and } x = 0. \end{aligned} \tag{4.6}$$

Proof. Let $q(x,t)$ denote the probability density of the stable Lévy process Y_t. An application of the Zolotarev duality formula for stable densities [5, Theorem 4.1] together with Lemma 3.1 shows that $\beta p(x,t) = q(x,t)$ for all $x > 0$ and $t > 0$, where $\beta = \mathbb{P}(Y_t > 0)$ for any $t > 0$. It follows from [26, Example 3.29] that the stable density $q(x,t)$ solves

$$\partial_t q(x,t) = D^\alpha_{-x} q(x,t) \tag{4.7}$$

for all $t > 0$ and all $x \in \mathbb{R}$. Hence we also have $\partial_t p(x,t) = D^\alpha_{-x} p(x,t)$ for $t > 0$ and $x > 0$. When $x = 0$, it follows from Proposition 4.1 and the continuity of $x \mapsto q(x,t)$ that $\beta p(0,t) = q(0,t)$ for $t > 0$. Then $\partial_t p(x,t) = D^\alpha_{-x} p(x,t)$ at $x = 0$ as well, since $D^\alpha_{-x} p(x,t)$ at $x = 0$ depends only on the values of $p(x,t)$ for $x \geqslant 0$.

It remains to verify the fractional boundary condition. From the general definition (4.5) it follows that for $1 < \alpha < 2$ we have

$$\begin{aligned} D^\alpha_{-x} f(x) &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_x^\infty f(y)(y-x)^{1-\alpha}\, dy, \\ D^{\alpha-1}_{-x} f(x) &= \frac{-1}{\Gamma(2-\alpha)} \frac{d}{dx} \int_x^\infty f(y)(y-x)^{1-\alpha}\, dy. \end{aligned} \tag{4.8}$$

Since $1 = \mathbb{P}(Z_t \geqslant 0)$ is constant for all $t > 0$, it follows that

$$\begin{aligned} 0 = \partial_t \mathbb{P}(Z_t \geqslant 0) &= \int_0^\infty \partial_t p(x,t)\, dx \\ &= \int_0^\infty D^\alpha_{-x} p(x,t)\, dx \\ &= \int_0^\infty \frac{d}{dx} \left(\frac{1}{\Gamma(2-\alpha)} \frac{d}{dx} \int_x^\infty p(y,t)(y-x)^{1-\alpha}\, dy \right) dx \\ &= -\int_0^\infty \frac{d}{dx} D^{\alpha-1}_{-x} p(x,t)\, dx \\ &= D^{\alpha-1}_{-x} p(0,t), \end{aligned} \tag{4.9}$$

since $\lim_{x\to\infty} D^{\alpha-1}_{-x} p(x,t) = \alpha \lim_{x\to\infty} D^{\alpha-1}_{-x} q(x,t) = 0$, by an elementary estimate using the fact that $q(x,t) \sim C x^{-\alpha-1}$ for some $C > 0$ as $x \to \infty$. $\square$

Let Z_t be the stochastic process defined in (3.3), where Y_t is a stable Lévy process with index $1 < \alpha < 2$ and characteristic function (3.2). Since $Z_t \geqslant 0$ by definition, it follows from Lemma 3.1 that Z_t is a conservative time-homogeneous Markov process whose (backward) semigroup $T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ is strongly continuous and contractive (i.e., $\|T_t f\| \leqslant \|f\|$) on the Banach space $X = C_\infty[0,\infty)$. Let A denote the generator of this semigroup, with domain $\mathrm{D}(A)$. Next we will show that this semigroup is also analytic and give a core for the generator. Recall that a core C_A of a closed linear operator A is a subset of its domain that is dense within the domain if the domain is equipped with the graph norm; i.e., C_A is a core of A if for each $f \in \mathrm{D}(A)$ there exists a sequence $\{f_n\} \subset C_A$ such that $f_n \to f$ and $Af_n \to Af$.

Write

$$S_b := \Big\{ f \in C_\infty[0,\infty) : f'' \in C(0,\infty),\ f''(x) = O(1) \text{ as } x \to \infty,$$
$$f''(x) = O(x^{\alpha-2}) \text{ as } x \to 0,\ f' \in C_b(0,\infty),\ f'(0+) = 0 \Big\} \tag{4.10}$$

and denote by ∂_x^α the (positive) Caputo fractional derivative of order $\alpha > 0$, which can be defined by

$$\partial_x^\alpha f(x) = I_x^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-y)^{n-1-\alpha} f^{(n)}(y)\, dy \tag{4.11}$$

where $n-1 < \alpha < n$ and $f^{(n)}$ is the nth derivative of f. The Caputo fractional derivative differs from the Riemann-Liouville form (4.4) because the operations of differentiation and (fractional) integration do not commute in general. For example, when $0 < \alpha < 1$ we have $\partial_x^\alpha f(x) = D_x^\alpha f(x) - f(0)x^{-\alpha}/\Gamma(1-\alpha)$ for suitably nice bounded functions (e.g., see [26, p. 39]).

Theorem 4.3. *Let Z_t denote the Feller process* (3.3) *where Y_t is a stable Lévy process with index $\alpha = 1/\beta \in (1,2)$ and characteristic function* (3.2)*, and let $T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ be the associated transition semigroup on $C_\infty[0,\infty)$ with generator A. Then $\{T_t\}_{t\geqslant 0}$ is analytic, $C_A = \{f \in S_b : \partial_x^\alpha f \in C_\infty[0,\infty)\}$ is a core of A and $Af = \partial_x^\alpha f$ for all $f \in C_A$, where ∂_x^α is the Caputo fractional derivative* (4.11).

Proof. It follows from [9, Proposition 4] that $C_A \subset \mathrm{D}(A)$ and $Af = \partial_x^\alpha f$ for all $f \in C_A$. Note that the extension from f'' bounded to $|f''(x)| = O(x^{\alpha-2})$ at $x = 0+$ is also mentioned in the proof.

Since A generates a strongly continuous contraction semigroup, the resolvent operators $R(\lambda, A) := (\lambda - A)^{-1}$ exist for all λ with $\operatorname{Re}\lambda > 0$ and they are bounded operators.

Let S_e be the span of $\{x \mapsto \exp(-cx) : c > 0\}$, which is dense in $C_\infty[0,\infty)$ by a standard Stone-Weierstraß argument. If we can show that $R(\lambda, A)S_e \subset C_A$ for some $\lambda > 0$, then C_A is a core of A.

Indeed, pick $f \in \mathrm{D}(A)$ and $g = \lambda f - Af$. Since S_e is dense in $C_\infty[0,\infty)$, there exists a sequence $\{g_n\} \subset S_e$ with $g_n \to g$. Thus, $f_n = R(\lambda, A)g_n \to f$ and $Af_n = \lambda f_n - g_n \to \lambda f - g = Af$. Since $f_n \in R(\lambda, A)S_e \subset C_A$, we see that C_A is a core.

It remains to show that $f_\lambda = R(\lambda, A)g$ is an element of C_A for any $g \in S_e$. The Laplace transform of a bounded and measurable function g is the unique analytic function defined by $\mathcal{L}g(s) = \int_0^\infty e^{-sx} g(x)\, dx$ for $\operatorname{Re} s > 0$; in particular, if $g(x) = e^{-cx}$ for some $c > 0$, its Laplace transform is given by $\mathcal{L}g(s) = 1/(s+c)$ for any s with $\operatorname{Re} s > 0$.

We will first define the function f_λ via its Laplace transform, and later we will show that in fact $f_\lambda = R(\lambda, A)g$. Let

$$\widehat{f_\lambda}(s) := \frac{\dfrac{1}{s+c} - \dfrac{s^{\alpha-1}\lambda^{1/\alpha-1}}{\lambda^{1/\alpha}+c}}{\lambda - s^\alpha}. \tag{4.12}$$

Next we show that $\widehat{f_\lambda}$ is the Laplace transform of a function $f_\lambda \in S_b$. Note that $\widehat{f_\lambda}$ can be extended to an analytic function on $\mathbb{C} \setminus (-\infty, 0]$, since the singularity at $s = \lambda^{1/\alpha}$ is removable. Furthermore, $s\widehat{f_\lambda}(s)$ is bounded on the complement of any neighborhood

of $s = -c$, and hence $\widehat{f_\lambda}$ is the Laplace transform of a bounded analytic function f_λ on $(0,\infty)$ in view of [2, Theorem 2.6.1] with $\omega = 0$. Using the Tauberian theorem [2, Theorem 2.6.4] we obtain that $\lim_{x\to\infty} f_\lambda(x) = \lim_{s\to 0} s\mathcal{L}f_\lambda(s) = 0$, $\lim_{x\to 0} f_\lambda(x) = \lim_{s\to\infty} s\mathcal{L}f_\lambda(s) = \lambda^{1/\alpha-1}/(\lambda^{1/\alpha}+c)$, and hence $f_\lambda \in C_\infty[0,\infty)$. Furthermore

$$\mathcal{L}f'_\lambda(s) = s\mathcal{L}f_\lambda(s) - f_\lambda(0) = \frac{\frac{s}{s+c} - \frac{s^\alpha \lambda^{1/\alpha-1}}{\lambda^{1/\alpha}+c}}{\lambda - s^\alpha} - \frac{\lambda^{1/\alpha-1}}{\lambda^{1/\alpha}+c} = \frac{\frac{s}{s+c} - \frac{\lambda^{1/\alpha}}{\lambda^{1/\alpha}+c}}{\lambda - s^\alpha}. \tag{4.13}$$

Using the Tauberian theorem again, $f'_\lambda(0) = 0$ and $f'_\lambda \in C_\infty[0,\infty)$. Taking the Laplace transform of $f_2 := f''_\lambda + \frac{c}{\lambda^{1/\alpha}+c}\frac{x^{\alpha-2}}{\Gamma(\alpha-1)}$ yields

$$\mathcal{L}f''_\lambda(s) + \frac{cs^{1-\alpha}}{\lambda^{1/\alpha}+c} = \frac{\frac{s^2}{s+c} - \frac{s\lambda^{1/\alpha}}{\lambda^{1/\alpha}+c}}{\lambda - s^\alpha} + \frac{cs^{1-\alpha}}{\lambda^{1/\alpha}+c} = \frac{cs^2 - cs\lambda^{1/\alpha}}{(s+c)(\lambda^{1/\alpha}+c)(\lambda - s^\alpha)} + \frac{cs^{1-\alpha}}{\lambda^{1/\alpha}+c}$$

and then the Tauberian theorem implies that $\lim_{x\to\infty} f_2(x) = 0$, $\lim_{x\to 0} f_2(x) = 0$. Hence $f_\lambda \in S_b$.

Next we show that $f_\lambda \in C_A$. Since the function $x \mapsto x^p \mathbb{1}_{[0,\infty)}(x)$ has Laplace transform $s^{-p-1}\Gamma(p+1)$ for any $p > -1$, it follows from (4.11) along with the convolution theorem for the Laplace transform that for $f \in S_b$ we have

$$\mathcal{L}[\partial_x^\alpha f](s) = s^{\alpha-2}\left(s^2\mathcal{L}f(s) - sf(0) - f'(0)\right) = s^\alpha \mathcal{L}f(s) - s^{\alpha-1}f(0).$$

Taking the Laplace transform of $\lambda f_\lambda - \partial_x^\alpha f_\lambda$, we therefore obtain

$$\begin{aligned}\lambda\mathcal{L}f_\lambda(s) - \mathcal{L}[\partial_x^\alpha f_\lambda](s) &= \lambda\mathcal{L}f_\lambda(s) - s^\alpha\mathcal{L}f_\lambda(s) + s^{\alpha-1}f_\lambda(0)\\ &= \frac{1}{s+c} - \frac{s^{\alpha-1}\lambda^{1/\alpha-1}}{\lambda^{1/\alpha}+c} + s^{\alpha-1}\frac{\lambda^{1/\alpha-1}}{\lambda^{1/\alpha}+c}\\ &= \frac{1}{s+c},\end{aligned} \tag{4.14}$$

which is the Laplace transform of g. Since f_λ and g are elements of $C_\infty[0,\infty)$, we see that $\partial_x^\alpha f_\lambda \in C_\infty[0,\infty)$ and hence $f_\lambda \in C_A$. Now it follows from the uniqueness of Laplace transforms that $f_\lambda = R(\lambda, A)g$. Therefore $R(\lambda, A)S_e \subset C_A$ and hence C_A is a core.

Finally we show that $\{T_t\}_{t\geqslant 0}$ is an analytic semigroup. Recall that the resolvent $R(\lambda, A)$ is a bounded operator for all $\operatorname{Re}\lambda > 0$. Then a general result from the theory of semigroups [2, Corollary 3.7.12] states that $\{T_t\}$ is bounded (i.e., for some $M > 0$ we have $\|T(t)\| \leqslant M$ in the operator norm for all $t \geqslant 0$) and analytic if $\|\lambda R(\lambda, A)\|$ is bounded in the right half plane; i.e. if there exists some $\tilde{M} > 0$ such that

$$\|\lambda R(\lambda, A)g\| \leqslant \tilde{M}\|g\| \tag{4.15}$$

for all $\operatorname{Re}\lambda > 0$ and all $g \in C_\infty[0,\infty)$, recalling that $\|g\| = \sup_{x\geqslant 0}|g(x)|$ is the Banach space norm. Note that it is actually enough to show (4.15) for g in a dense subset (such as S_e) due to the continuity of $R(\lambda, A)$.

Let $g \in S_e$, and recall that the Laplace transform

$$\mathcal{L}[R(\lambda, A)g](s) = \int_0^\infty e^{-sx}R(\lambda, A)g(x)\,dx = \frac{\mathcal{L}g(s) - \frac{s^{\alpha-1}}{\lambda^{1-1/\alpha}}\mathcal{L}g(\lambda^{1/\alpha})}{\lambda - s^\alpha} \tag{4.16}$$

can be extended analytically (and uniquely) to $s \in \mathbb{C} \setminus (-\infty, 0]$ for any $\operatorname{Re}\lambda > 0$. Then for $x > 0$, the complex inversion formula for Laplace transforms [34, Theorem II.7.4] implies that

$$[R(\lambda, A)g](x) = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{a-iN}^{a+iN} e^{sx} \frac{\mathcal{L}g(s) - \frac{s^{\alpha-1}}{\lambda^{1-1/\alpha}} \mathcal{L}g(\lambda^{1/\alpha})}{\lambda - s^\alpha} \, ds \tag{4.17}$$

independent of $a > 0$. For $g \in S_e$, the integrand in (4.17) is bounded and analytic on an open neighbourhood of the set $\{s \in \mathbb{C} : 0 \leqslant \operatorname{Re}(s) \leqslant a, |s| > r\}$ for any $0 < r < a$. Then a standard argument using the Cauchy integral theorem shows that (4.17) also holds for $a = 0$. Substituting $i\xi = s$, we have

$$\begin{aligned} [R(\lambda, A)g](x) &= \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\xi x} \left(\frac{\mathcal{L}g(i\xi)}{\lambda - (i\xi)^\alpha} - \frac{\frac{(i\xi)^{\alpha-1}}{\lambda^{1-1/\alpha}} \mathcal{L}g(\lambda^{1/\alpha})}{\lambda - (i\xi)^\alpha} \right) d\xi \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\xi x} \left(\frac{\mathcal{L}g(i\xi)}{\lambda - (i\xi)^\alpha} - \frac{(i\xi)^{\alpha-1}}{\lambda - (i\xi)^\alpha} \frac{\mathcal{L}g(\lambda^{1/\alpha})}{\lambda^{1-1/\alpha}} \right) d\xi \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\xi x} I_\lambda^1(\xi) \, d\xi - \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\xi x} I_\lambda^2(\xi) \, d\xi \frac{\mathcal{L}g(\lambda^{1/\alpha})}{\lambda^{1-1/\alpha}}, \end{aligned} \tag{4.18}$$

for any $\operatorname{Re}\lambda > 0$. Observe that

$$\frac{1}{\lambda - (i\xi)^\alpha} = \int \int_0^\infty e^{-i\xi x} e^{-\lambda t} \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt \, dx \tag{4.19}$$

is the Fourier-Laplace transform of the marginal densities of an α-stable process, where $\int e^{-i\xi x} g_\alpha(x) \, dx = \exp((i\xi)^\alpha)$. They are also the Green's functions (convolution kernel) for the bounded analytic convolution semigroup generated by D_x^α on $L_1(\mathbb{R})$ and also, by the transference principle in [4, Theorem 4.6 and Corollary 4.2], on $C_\infty(\mathbb{R})$. Hence the resolvent (the Laplace transform of the semigroup) satisfies

$$R(\lambda, D_x^\alpha)g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} I_\lambda^1(\xi) \, d\xi$$

and then [2, Corollary 3.7.12] implies that $\|\lambda R(\lambda, D_x^\alpha)g\| \leqslant M\|g\|$ for some $M > 0$, for all $\operatorname{Re}(\lambda) > 0$.

In order to estimate the term involving I_λ^2, observe that, since g_α is an α-stable density, $g_\alpha(x)$ is differentiable, $g_\alpha(x) \sim x^{-\alpha-1}$ as $x \to \infty$ and $g_\alpha(x)$ decays super-exponentially as $x \to -\infty$; hence the function

$$F_\lambda(x) := \int_0^\infty e^{-\lambda t} \frac{x}{t} \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt$$

is Lebesgue-integrable in x and differentiable for $x > 0$. Use (4.19) to see that

$$I_\lambda^2(\xi) = \frac{-i}{\alpha} \int_\lambda^\infty \frac{d}{d\xi} \left[\frac{1}{\mu - (i\xi)^\alpha} \right] d\mu = -\frac{1}{\alpha} \int e^{-i\xi x} F_\lambda(x) \, dx.$$

Since $I_\lambda^2(\xi)$ is the Fourier transform of an integrable function which is also differentiable, the Fourier inversion theorem [36, Theorem7.2-1] yields

$$\frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N} e^{i\xi x} I_\lambda^2(\xi)\,d\xi = -\frac{1}{\alpha}F_\lambda(x)$$

for all $x > 0$. Since $|I_\lambda^2(\xi)| \sim |\xi|^{-1}$ as $|\xi| \to \infty$, one needs the Cauchy principal value here. A simple change of variable then yields

$$\left|-\frac{1}{\alpha}F_\lambda(x)\right| = \left|\int_0^\infty e^{-\lambda t}\frac{x}{\alpha t^{1+1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right)\,dt\right| = \left|\int_0^\infty e^{-\lambda\left(\frac{x}{u}\right)^\alpha} g_\alpha(u)\,du\right| \leqslant 1$$

for all $\mathrm{Re}(\lambda) > 0$ and $x > 0$. Since $|\lambda^{1/\alpha}| \leqslant \mathrm{Re}(\lambda^{1/\alpha})/\cos(\pi/2\alpha)$ for $\mathrm{Re}(\lambda) > 0$, and $\mathrm{Re}(\lambda|\mathcal{L}h(\lambda)|) \leqslant \|h\|_\infty$ for any $h \in C_\infty[0,\infty)$, it follows that

$$\left|\lambda\frac{\mathcal{L}g(\lambda^{1/\alpha})}{\lambda^{1-1/\alpha}}\right| = \left|\lambda^{1/\alpha}\mathcal{L}g(\lambda^{1/\alpha})\right| \leqslant \frac{1}{\cos(\pi/2\alpha)}|\,\mathrm{Re}(\lambda^{1/\alpha})\mathcal{L}g(\lambda^{1/\alpha})| \leqslant \frac{1}{\cos(\pi/2\alpha)}\|g\|$$

and hence

$$\|\lambda R(\lambda, A)g\| \leqslant \left(M + \frac{1}{\cos(\pi/2\alpha)}\right)\|g\|$$

for all $\mathrm{Re}\,\lambda > 0$. Then it follows from [2, Corollary 3.7.12] that $\{T_t\}_{t\geqslant 0}$ is analytic. $\square$

Remark 4.4. Patie and Simon [29] show that the reflected stable process Z_t in Theorem 4.3 has the backward generator

$$Af(x) = f'(0)\frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^x f''(x-y)\frac{y^{1-\alpha}}{\Gamma(2-\alpha)}\,dy. \tag{4.20}$$

They also give the exact domain of the generator [29, Proposition 2.2]. If $f \in S_b$, then $f'(0) = 0$, and Af reduces to the Caputo fractional derivative (4.11).

In view of Theorem 4.3, $T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ is a strongly continuous, analytic semigroup on the Banach space $X := C_\infty[0,\infty)$ with the supremum norm, with generator $Af(x) = \partial_x^\alpha f(x)$ for $f \in S_b$ such that $\partial_x^\alpha f \in X$. The dual (or adjoint) semigroup T_t^* is defined on the dual space $C^*[0,\infty) = \mathcal{M}_b[0,\infty)$ of finite signed Radon measures on $[0,\infty)$ equipped with the total variation norm: Given a measure $\mu \in \mathcal{M}_b[0,\infty)$, use the Jordan decomposition to write $\mu = \mu^+ - \mu^-$ uniquely as a difference of two positive measures, and define $\|\mu\| = \mu^+[0,\infty) + \mu^-[0,\infty)$. The dual semigroup satisfies

$$\int T_t f(x)\mu(dx) = \int f(x)[T_t^*\mu](dx) \tag{4.21}$$

for all $f \in C_\infty[0,\infty)$ and all $\mu \in \mathcal{M}_b[0,\infty)$. See [14, Section 2.5] for more details. In probabilistic terms, since $T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ for this time-homogeneous Markov process, equation (4.21) implies that

$$\begin{aligned}\int T_t f(x)\mu(dx) &= \int \mathbb{E}[f(Z_{t+s})|Z_s = x]\mu(dx)\\ &= \int f(y)P_t(dy,\mu) = \int f(y)[T_t^*\mu](dy)\end{aligned}$$

where $P_t(y,\mu) = \int P(y,x,t)\mu(dx)$ and $P(y,x,t) = \mathbb{P}[Z_{t+s} \leqslant y | Z_s = x]$ is the transition probability distribution of the Markov process Z_t. Hence, if μ is the probability distribution of Z_s, then $T_t^*\mu(dy) = P_t(dy,\mu)$ is the probability distribution of Z_{t+s}. The dual semigroup is also called the *forward semigroup* associated with the Markov process Z, since it maps the probability distribution forward in time.

Next we will compute the generator A^* of the forward semigroup. This is the adjoint of the generator A of the backward semigroup, in the sense that

$$\int Af(x)\mu(dx) = \int f(x)[A^*\mu](dx)$$

for all $f \in \mathrm{D}(A)$ and $\mu \in \mathrm{D}(A^*)$. Theorem 4.5 will show that every measure $\mu \in \mathrm{D}(A^*)$ has a Lebesgue density $g \in L^1[0,\infty)$, so that $\mu(dy) = g(y)\,dy$, and that the adjoint $A^*g := A^*\mu$ of the positive fractional Caputo derivative $Af(x) = \partial_x^\alpha f(x)$ in our setting is the negative Riemann-Liouville fractional derivative $A^*g(y) = D_{-y}^\alpha g(y)$ using (4.5).

The forward semigroup T_t^* of a Markov process is not, in general, strongly continuous on $\mathcal{M}_b[0,\infty)$. That is, there exist measures μ such that $T_t^*\mu \not\to \mu$ in the total variation norm as $t \downarrow 0$. For example, if T_t^* is the forward semigroup associated with the diffusion equation $\partial_t p = \partial_x^2 p$, and $\mu = \delta_0$ is a point mass at the origin, then $T_t^*\mu$ is a Gaussian probability measure with mean 0 and variance $2t$ for all $t > 0$, and since $\mu\{0\} = 1$ and $T_t\mu\{0\} = 0$ for all $t > 0$, we have $\|T_t\mu - \mu\| = 1$ for all $t > 0$ in the total variation norm.

To handle this situation, we introduce the *sun dual space* of $X := C_\infty[0,\infty)$:

$$X^\odot := \{\mu \in X^* : \lim_{t\downarrow 0} \|T_t^*\mu - \mu\| = 0\}$$

is closed subspace of $X^* = \mathcal{M}_b[0,\infty)$ on which the forward semigroup is strongly continuous. It follows from basic semigroup theory [14, Section 2.6] that for $\mu \in X^\odot$, $T_t^*\mu \in X^\odot$ for all $t \geqslant 0$, and $X^\odot = \overline{\mathrm{D}(A^*)}$. The restriction of $\{T_t^*\}_{t\geqslant 0}$ to $X^\odot$ is called the *sun dual semigroup* $\{T_t^\odot\}_{t\geqslant 0}$ with generator $A^\odot\mu = A^*\mu$ for all $\mu \in \mathrm{D}(A^\odot)$, where

$$\mathrm{D}(A^\odot) = \{\mu \in \mathrm{D}(A^*) : A^*\mu \in X^\odot\}. \tag{4.22}$$

For the reflected stable process, we will show in Theorem 4.5 that $C_\infty^\odot[0,\infty)$ is the space of absolutely continuous elements of $\mathcal{M}_b[0,\infty)$,

$$\mathcal{M}_{ac}[0,\infty) = \big\{\mu \in \mathcal{M}_b[0,\infty)\ :\ \mu(dy) = g(y)\,dy,\ \exists\, g \in L^1[0,\infty)\big\}.$$

and we will derive the forward equation of the reflected stable process on the sun-dual space. For a general bounded measure $\mu \in \mathcal{M}_b[0,\infty)$, we will then prove in Corollary 4.8 that $T_t^*\mu$ can be computed as the vague limit of $T_t^\odot\mu_n$, where $\mu_n \to \mu$ vaguely, and $\mu_n \in C_\infty^\odot[0,\infty)$ for all n.

Theorem 4.5. *Let Z_t denote the Feller process* (3.3)*, where Y_t is a stable Lévy process with index $\alpha = 1/\beta \in (1,2)$ and characteristic function* (3.2)*, with (backward) semigroup $T_tf(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ on $C_\infty[0,\infty)$. Then $C_\infty^\odot[0,\infty) = \mathcal{M}_{ac}[0,\infty)$ and the generator $A^\odot g := A^\odot\mu$ of the sun-dual semigroup $\{T^\odot(t)\}_{t\geqslant 0}$ is given by*

$$A^\odot g(y) = D_{-y}^\alpha g(y) \tag{4.23}$$

with domain $\mathrm{D}(A^\odot) = \{g \in L^1[0,\infty)\ :\ D_{-y}^\alpha g(y) \in L^1[0,\infty), D_{-y}^{\alpha-1} g(0) = 0\}$.

Proof. Suppose $A^*\mu = \nu \in \mathcal{M}_b[0,\infty)$ for some $\mu \in \mathcal{M}_b[0,\infty)$, so that $\int Af(x)\mu(dx) = \int f(x)\nu(dx)$ for all $f \in \mathrm{D}(A)$. Set $v(x) := \nu[0,x]$ for $x \geqslant 0$ and $v(x) = 0$ for $x < 0$. If $f \in S_b$ with $\partial_x^\alpha f \in C_\infty[0,\infty)$, then it follows from Theorem 4.3 that $f \in \mathrm{D}(A)$ and $Af = \partial_x^\alpha f$. It is obvious from the Definition (4.11) that $\partial_x^{\alpha-1} f'(x) = \partial_x^\alpha f(x)$. Let

$$I_x^\alpha f(x) = \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} f(y)\,dy$$

denote the positive Riemann-Liouville fractional integral (4.2) of order $\alpha > 0$ for a function $f \in C_\infty[0,\infty)$, and apply the general formula [6, Eq. (1.21)] $I_x^{\alpha-1}\partial_x^{\alpha-1} f'(x) = f'(x) - f'(0)$ to see that $f'(x) = I_x^{\alpha-1}\partial_x^{\alpha-1} f'(x) = I_x^{\alpha-1}\partial_x^\alpha f(x) = I_x^{\alpha-1} Af(x)$. Since $f(x) \to 0$ as $x \to \infty$, and $v(0) = 0$ for $x < 0$, we can apply the integration by parts formula [17, Theorem 19.3.13]

$$\int_a^b f(x)\nu(dx) = f(b)v(b) - f(a)v(a) - \int_a^b v(x) f'(x) dx$$

with $a < 0$, and then let $b \to \infty$, to see that

$$\int_0^\infty f(x)\nu(dx) = -\int_0^\infty f'(x) v(x)\,dx.$$

Thus, for all $f \in S_b$ with $\partial_x^\alpha f \in C_\infty[0,\infty)$, a Fubini argument yields

$$\begin{aligned} \int_0^\infty f(x)\nu(dx) &= -\int_0^\infty \int_0^x \frac{(x-y)^{\alpha-2}}{\Gamma(\alpha-1)} Af(y)\,dy\, v(x)\,dx \\ &= -\int_0^\infty \int_y^\infty \frac{(x-y)^{\alpha-2}}{\Gamma(\alpha-1)} v(x)\,dx\, Af(y)\,dy \\ &= \int_0^\infty Af(y)\,\mu(dy). \end{aligned} \tag{4.24}$$

Next we will show that $S := \{Af : f \in S_b \text{ with } \partial_x^\alpha f \in C_\infty[0,\infty)\}$ is dense in $C_\infty[0,\infty)$, and then it will follow that any measure $\mu \in \mathrm{D}(A^*)$ has a Lebesgue density

$$g(y) = -\int_y^\infty \frac{(x-y)^{\alpha-2}}{\Gamma(\alpha-1)} v(x)\,dx = -I_{-y}^{\alpha-1} v(y) \tag{4.25}$$

where $A^*\mu = \nu$ and $v(x) = \nu[0,x]$. Let $C_c^\infty[0,\infty)$ denote the space of smooth functions with compact support, i.e., such that $h(x) = 0$ for all $x > M$, for some $M > 0$. It is not hard to check that the space $Q = \{h \in C_c^\infty[0,\infty) : \int h = 0\}$ is dense in $C_\infty[0,\infty)$. Then we certainly have

$$\lim_{x\to\infty} I_x^\alpha h(x) = \lim_{x\to\infty} \int_0^M \frac{(x-s)^{\alpha-1} - x^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds = 0,$$

and therefore, the function $f(x) = I_x^\alpha h(x)$ is an element of $C_\infty[0,\infty)$ for any $h \in Q$. Elementary estimates suffice to check that $f \in S_b$ as well. Since the positive Caputo derivative is a left inverse of the positive Riemann-Liouville integral [6, Eq. (1.21)], we also have $\partial_x^\alpha f(x) = \partial_x^\alpha I_x^\alpha h(x) = h(x) \in C_\infty[0,\infty)$. Hence $h = Af \in S$ for all $h \in Q$, and thus $Q \subseteq S$. Now for any $f \in C_\infty[0,\infty)$ there exists a sequence $f_n \to f$ in the supremum norm, with $f_n \in S$ for all n. Then some simple estimates can be used to

verify that $\int f(y)\mu(dy) = \int f(y)g(y)\,dy$, and it follows that the measure μ has the Lebesgue density g. Hence we can identify $\mathrm{D}(A^*)$ with a subspace of $L^1[0,\infty)$.

Next we will show that this subspace is dense in $L^1[0,\infty)$. Define

$$\phi_n(x) := \tfrac{1}{\Gamma(\alpha)}\left(\tfrac{1}{n}-x\right)^{\alpha-1} \mathbb{1}_{[0,1/n)}(x),$$

and note that $D^{\alpha-1}_{-x}\phi_n(x)\equiv 1$ for $x\in(0,1/n)$ by a straightforward computation. Note that the set $U := \{g(x) - [D^{\alpha-1}_{-x}g(0)]\phi_n : g\in C_c^\infty[0,\infty), n\in\mathbb{N}\}$ is dense in $L^1[0,\infty)$, and that $D^\alpha_{-x}g(x)\in L^1[0,\infty)$ for all $g\in U$. Furthermore, U is a subset of

$$G = \left\{g\in L^1[0,\infty) \,:\, D^\alpha_{-x}g\in L^1[0,\infty),\ D^{\alpha-1}_{-x}g(0)=0\right\}.$$

For any $g\in G$ and $f\in S_b$ with $\partial^\alpha_x f\in C_\infty[0,\infty)$, we have $D^{\alpha-1}_{-x}g(0)=0$ and $f'(0)=0$. Then Theorem 4.3, a Fubini argument, and integration by parts (twice) using equation (4.5) yields

$$\begin{aligned}
\int_0^\infty Af(y)\,g(y)\,dy &= \int_0^\infty\int_0^y \frac{(y-x)^{1-\alpha}}{\Gamma(2-\alpha)}f''(x)\,dx\,g(y)\,dy\\
&= \int_0^\infty\int_x^\infty \frac{(y-x)^{1-\alpha}}{\Gamma(2-\alpha)}g(y)\,dy\,f''(x)\,dx\\
&= \int_0^\infty I^{2-\alpha}_{-x}g(x)f''(x)\,dx\\
&= \int_0^\infty D^{\alpha-1}_{-x}g(x)f'(x)\,dx\\
&= \int_0^\infty f(x)\,D^\alpha_{-x}g(x)\,dx.
\end{aligned} \tag{4.26}$$

Furthermore, as $\{f\in S_b : Af\in C_\infty[0,\infty)\}$ is a core, for all $f\in\mathrm{D}(A)$ there exists a sequence f_n in the core such that $f_n\to f$ and $Af_n\to Af$. Hence (4.26) holds for all $f\in\mathrm{D}(A)$ and therefore $g\in\mathrm{D}(A^*)$ and $A^*g = D^\alpha_{-x}g$ for any $g\in G$. Since U is dense in $L^1[0,\infty)$, and $U\subseteq G\subseteq\mathrm{D}(A^*)$, it follows that $\mathrm{D}(A^*)$ is dense in $L^1[0,\infty)$. Since $C^\odot_\infty[0,\infty)$ is the smallest closed set containing $\mathrm{D}(A^*)$ by definition, and since $L^1[0,\infty)$ is a closed subspace of $\mathcal{M}_b[0,\infty)$, we have shown that $L^1[0,\infty) = C^\odot_\infty[0,\infty)$.

Equation (4.22) implies that $\nu = A^*\mu$ is an element of $C^\odot_\infty[0,\infty)$ for any $\mu\in\mathrm{D}(A^\odot)$, and therefore, we have $\nu(dx) = h(x)\,dx$ for some $h\in L^1[0,\infty)$, as well as $\mu(dy) = g(y)\,dy$ for some $g\in L^1[0,\infty)$. Since $v(x)=\nu[0,x]$, it follows that $h(x)=v'(x)$. Since $D^{\alpha-1}_{-y}$ is a left inverse of $I^{\alpha-1}_{-y}$ in general, it follows from (4.25) that $v(y) = -D^{\alpha-1}_{-y}g(y)$. Then we have

$$h(x) = v'(x) = \frac{d}{dx}\left[-D^{\alpha-1}_{-x}g(x)\right] = D^\alpha_{-x}g(x) = A^*g(x) \tag{4.27}$$

for all $g\in\mathrm{D}(A^\odot)$, which proves the generator formula (4.23). Equation (4.27) also shows that $D^\alpha_{-x}g(x)\in L^1[0,\infty)$ for all $g\in\mathrm{D}(A^\odot)$, and since v is continuous with $v(0)=0$, it follows that $D^{\alpha-1}_{-x}g(0)=v(0)=0$ for all $g\in\mathrm{D}(A^\odot)$. This proves that $\mathrm{D}(A^\odot)\subseteq G$. Since $\mathrm{D}(A^\odot)$ is defined as the set of $g\in\mathrm{D}(A^*)$ such that $A^*g\in L^1[0,\infty)$, it follows from (4.26) that $g\in\mathrm{D}(A^\odot)$ for all $g\in G$, so that $G\subseteq\mathrm{D}(A^\odot)$ as well, which completes the proof. $\square$

Theorem 4.5 establishes the forward equation of the reflected stable process Z_t, for certain initial conditions. It shows that, for any initial condition $\mu_0(dx) = g(x)dx$ where $g \in L^1[0,\infty)$, $\mu_t := T^{\odot}\mu$ solves the Cauchy problem

$$\partial_t \mu(t) = A^{\odot}\mu(t); \quad \mu(0) = \mu_0$$

on the sun-dual space. This implies that, for any probability density $p_0(x)$ such that $p_0(x) = 0$ for $x < 0$, the function $p(x,t) = T^{\odot}p_0(x)$ solves the forward equation

$$\partial_t p(x,t) = D^{\alpha}_{-x}p(x,t); \quad p(x,0) = p_0(x), \quad D^{\alpha-1}_{-x}p(x,t)\Big|_{x=0} \equiv 0. \tag{4.28}$$

Remark 4.6. Proposition 4.1 implies that the probability distribution of Z_t started at $Z_0 = 0$ has a density function $p(x,t)$ given by (4.1). Fix $s > 0$ and take $p_0(x) = p(x,s)$. Then (4.28) holds for all $t > s$. Since $s > 0$ is arbitrary, this can be used to give an alternative proof of Theorem 4.2.

The next two results will allow us to compute the transition probability density $y \mapsto p(x,y,t)$ of the time-homogeneous Markov process $y = Z_{t+s}$ for any initial state $x = Z_s$, by applying Theorem 4.5 to a sequence of initial conditions $\mu_n \in X^{\odot}$ such that $\mu_n \to \delta_x$. The first result shows that the transition density exists.

Corollary 4.7. *The semigroup $T^{\odot}_t$ is a strongly continuous bounded analytic semigroup on $L^1(\mathbb{R}_+)$ and the transition probability distributions $T^*_t\delta_x$ have smooth densities $y \mapsto p(x,y,t)$ for all $t > 0$ and all $x \geqslant 0$.*

Proof. It is well-known that the spectra of A and A^* coincide, and $R(\lambda, A^*) = R(\lambda, A)^*$ for all λ in the resolvent set of A (and A^*). Therefore, since A is a sectorial operator, being the generator of a bounded analytic semigroup, it follows that A^* is a sectorial operator as well, and hence A^* generates a bounded analytic semigroup (not necessarily strongly continuous at 0) on $C^*_\infty[0,\infty) = \mathcal{M}_b[0,\infty)$ by [2, Theorem 3.7.1] which coincides with T^*_t for all $t > 0$ by the uniqueness of the Laplace transform. Therefore, the restriction of A^* to $\overline{\mathrm{D}(A^*)} = L^1[0,\infty)$ (i.e., the operator $A^{\odot}$) generates a strongly continuous bounded analytic semigroup $T^{\odot}_t$ on $L^1[0,\infty)$ by [2, Remark 3.7.13]. Furthermore, [2, Remark 3.7.20] shows that for all $n \in \mathbb{N}$ and $t > 0$ we have that $T^*_t\mu \in \mathrm{D}((A^*)^n)$ for all $\mu \in \mathcal{M}_b[0,\infty)$ and that $T^{\odot}_t f \in \mathrm{D}((A^{\odot})^n)$ for all $f \in L^1[0,\infty)$. Since $\mathrm{D}(A^*) \subset L^1[0,\infty)$, it follows that $T^*_s\mu \in L^1[0,\infty)$ for all $s > 0$ and $\mu \in \mathcal{M}_b[0,\infty)$. Therefore for all $t > 0$ and $\mu \in \mathcal{M}_b[0,\infty)$ we have

$$T^*_t\mu = T^*_{\frac{t}{2}}T^*_{\frac{t}{2}}\mu = T^{\odot}_{\frac{t}{2}}T^*_{\frac{t}{2}}\mu \in \mathrm{D}((A^{\odot})^n)$$

for all $n \in \mathbb{N}$. Thus by taking $\mu = \delta_x$ it follows that $T^*_t\delta_x \in \mathrm{D}((A^{\odot})^n)$ for all n. Using Theorem 4.5, we have that $\mathrm{D}(A^{\odot}) = \{g \in L^1[0,\infty) : D^{\alpha}_{-y}g(y) \in L^1[0,\infty), D^{\alpha-1}_{-y}g(0) = 0\}$, and then it follows that $\left(D^{\alpha}_{-y}\right)^n T^*_t\delta_x \in L^1[0,\infty)$ for all n. In particular, the transition probability distribution $T^*_t\delta_x$ has a density function $y \mapsto p(x,y,t)$ for all $t > 0$ and all $x \geqslant 0$. To see that this function is smooth note that any positive integer m can be written in the form $m = n\alpha - \beta$ for some integer $n \geqslant 1$ and some positive real number $\beta < \alpha$. A straightforward calculation shows that for $f \in \mathrm{D}((A^{\odot})^n)$ we have $\left(D^{\alpha}_{-y}\right)^n f = D^{n\alpha}_{-y}f$ and $I^{\beta}_{-y}D^{n\alpha}_{-y}f = D^{n\alpha-\beta}_{-y}f \in L^1[0,\infty)$ for all $\beta < \alpha$ and $n \geqslant 1$. Since $D^m_{-y}f = (-1)^m(d/dy)^m f$, we have $(d/dy)^m p(x,y,t) \in L^1[0,\infty)$ for all m. $\square$

Corollary 4.8. *Let* $\{\mu_n\} \subset C_\infty^\odot[0,\infty) = \mathcal{M}_{ac}[0,\infty)$ *such that* $\mu_n \to \mu$ *vaguely as* $n \to \infty$ *for some* $\mu \in \mathcal{M}_b[0,\infty)$*, then* $T_t^\odot \mu_n \to T_t^* \mu$ *vaguely as* $n \to \infty$.

Proof. Since $T_t^\odot \mu_n = T_t^* \mu_n$ for all $n \geqslant 1$, for all $\phi \in C_\infty[0,\infty)$ we have

$$\begin{aligned} \int \phi(x)[T_t^\odot \mu_n](dx) = \int \phi(x)[T_t^* \mu_n](dx) = \int [T_t\phi](x)\mu_n(x)dx \\ \xrightarrow{n\to\infty} \int [T_t\phi](x)\mu(dx) = \int \phi(x)[T_t^*\mu](dx), \end{aligned} \tag{4.29}$$

and the result follows. □

In Section 5, we will apply Corollary 4.8 to compute the transition densities of the reflected stable process to any desired degree of accuracy, by solving the forward equation numerically. For any initial state $Z_s = x$, we will approximate the initial condition $\mu_0 = \delta_x$ in the numerical method by a sequence of measures μ_n with L^1-densities, and then Corollary 4.8 guarantees that the resulting solutions converge to the transition density in the supremum norm as $n \to \infty$.

Remark 4.9. The calculation (4.26) of the forward generator (4.23) uses the $L^2[0,\infty)$ adjoint formula

$$\int_0^\infty f(x)A^*g(x)\,dx = \int_0^\infty Af(x)g(x)\,dx.$$

The negative Riemann-Liouville fractional derivative $A^* = D_{-x}^\alpha$ is the $L^2(\mathbb{R})$ adjoint of the positive Riemann-Liouville fractional derivative $A = D_x^\alpha$ (e.g., see [37]), but the adjoint calculation has to be modified on the half-line, using the boundary condition at the origin. This boundary condition is what distinguishes the forward equation (4.28) of the reflected stable process Z_t in (3.3) from the forward equation (4.7) of the original stable process Y_t, since both have the same forward generator $A^* = D_{-x}^\alpha$. This is a natural extension of the special case $\alpha = 2$, where Z_t is a reflected Brownian motion.

Remark 4.10. Using integration by parts, one can write the backward generator in the form of an integro-differential operator (e.g., see Jacob [19])

$$Af(x) = b(x)f'(x) + \int \left[f(x+y) - f(x) - yf'(x)\right]\phi(x,dy), \tag{4.30}$$

with coefficients

$$\begin{aligned} b(x) &= \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} \quad \text{and} \\ \phi(x,dy) &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}|y|^{-1-\alpha}dy\mathbb{1}_{(-x,0)}(y) + \frac{\alpha-1}{\Gamma(2-\alpha)}x^{-\alpha}\varepsilon_{-x}(dy). \end{aligned} \tag{4.31}$$

The jump intensity $\phi(x,dy)$ describes the behavior of the process Z_t, which truncates jumps of the stable process Y_t starting at the point $x > 0$ in the state space, so that a jump (they are all negative) of size $|y| > x$ is changed to a jump of size x. This keeps the sample paths of Z_t inside the half-line $[0,\infty)$. Since the drift $b(x)$ is unbounded, the existence of a Markov process Z_t with generator (4.30) would not follow from general theory (e.g., see [15, Section 4.5] or [32, Section 3]). Hence the reflected stable process Z_t is an interesting example of a Markov process with unbounded drift coefficient.

5. Transition density of the reflected stable process

For the reflected stable process Z_t started at $Z_0 = 0$, the form of the transition density was established in Proposition 4.1. To the best of our knowledge, there is no known analytical formula for the transition density $y \mapsto p(x;t,y)$ of the reflected stable process $y = Z_{t+s}$ started at $x = Z_s > 0$. In this section, we compute and plot this transition density, by numerically solving the associated forward equation (4.28). The existence and smoothness of these transition densities is guaranteed by Corollary 4.7. Corollary 4.8 shows that $T_t^{\odot} g_n(y)dy \to T_t^* \delta_x(dy)$ in the supremum norm as $n \to \infty$ for any sequence of functions $g_n \in L^1[0,\infty)$ such that $g_n(y)dy \to \delta_x(y)dy$ (vague convergence). We take $g_n(y) = n\, \mathbb{1}_{[x,x+1/n]}(y)$. Then the solutions $T_t^{\odot} g_n$ provide estimates of the transition density to any desired degree of accuracy. Theorem 4.5 shows that $T_t^{\odot} g_n$ solves the forward equation (4.28). Hence, we can compute the transition densities of the stable process by solving the forward equation numerically, with this initial condition.

In order to compute the probability density $p(x,y,t)$ numerically, we consider the fractional boundary value problem

$$\partial_t u(y,t) = D_{-y}^{\alpha} u(y,t); \quad u(y,0) = \delta_x(y); \quad D_{-y}^{\alpha-1} u(0,t) = 0. \tag{5.1}$$

We develop forward-stepping numerical solutions $u_h(y_i,t)$ that estimate $u(y_i,t)$ at locations $y_i = ih$ for $i = 0,1,\ldots,N$ over an interval $[0, y_{max}]$ in the state space, where y_{max} is chosen large enough and h is chosen small enough so that enlarging the domain further, or making the step size smaller, has no appreciable effect on the computed solutions (e.g., the resulting graph does not visibly change). We approximate the delta function initial condition $u(y,0) = \delta_x(y)$ by setting $u_h(y_i,0) = 1/h$ for $y_i = x$ and $u_h(y_i,0) = 0$ otherwise, a numerical representation of the initial condition $g_n(y) = n\, \mathbb{1}_{[x,x+1/n]}(y)$ with $h = 1/n$.

Numerical methods for fractional differential equations are an active area of research. One important finding [23, 24] is that a shifted version of the Grünwald finite difference formula (5.6) for the fractional derivative is required to obtain a stable, convergent method. Hence we approximate

$$D_{-y}^{\alpha} u(y_i,t) \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{N+1-i} w_k^{\alpha} u_h(y_{i+k-1},t) \quad \text{where} \quad w_k^{\alpha} := (-1)^k \binom{\alpha}{k} \tag{5.2}$$

for $i \geqslant 2$. Note that this approximation of $D_{-y}^{\alpha} u(y_i,t)$ does not depend on the value of u_h at the boundary $y_0 = 0$. We enforce the boundary condition at $y_0 = 0$ at each step; i.e., we set

$$u_h(y_0,t) = -\sum_{k=1}^{N} w_k^{\alpha-1} u_h(y_k,t)$$

so that

$$D_{-y}^{\alpha-1} u(y_0,t) \approx \sum_{k=0}^{N} w_k^{\alpha-1} u_h(y_k,t) = 0$$

since $w_0^{\alpha-1}=1$. Finally, for $i=1$ we approximate

$$\begin{aligned}
D^{\alpha}_{-y}u(y_1,t) \approx & \frac{1}{h^{\alpha}}\sum_{k=0}^{N} w_k^{\alpha} u_h(y_{1+k-1},t)\\
= & \frac{1}{h^{\alpha}}\left(\sum_{k=1}^{N} w_k^{\alpha} u_h(y_k,t) - \sum_{k=1}^{N} w_k^{\alpha-1} u_h(y_k,t)\right)\\
= & -\frac{1}{h^{\alpha}}\sum_{k=1}^{N} w_{k-1}^{\alpha-1} u_h(y_k,t)
\end{aligned} \tag{5.3}$$

using an elementary identity for fractional binomial coefficients $w_k^{\alpha}-w_k^{\alpha-1}=-w_{k-1}^{\alpha-1}$.

This leads to the following linear system of ordinary differential equations,

$$\frac{d}{dt}\begin{pmatrix} u_h(y_1,t)\\ \vdots \\ \vdots \\ \vdots \\ u_h(y_N,t)\end{pmatrix} = \frac{1}{h^{\alpha}}\begin{pmatrix} -1 & \alpha-1 & \dots & \dots & -w_{N-1}^{\alpha-1}\\ 1 & -\alpha & w_2^{\alpha} & \dots & w_{N-1}^{\alpha}\\ 0 & 1 & \ddots & \ddots & w_{N-2}^{\alpha}\\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -\alpha \end{pmatrix}\begin{pmatrix} u_h(y_1,t)\\ \vdots \\ \vdots \\ \vdots \\ u_h(y_N,t)\end{pmatrix} \tag{5.4}$$

whose solution can be approximated using any numerical ODE solver (indeed, this is a linear system of the form $u'=Au$, so it has a solution $u(t)=e^{tA}u(0)$ for any initial condition). The resulting numerical solutions with $y_{\max}=12$, $h=0.01$, starting points $x=0,1,2,4$, and $\alpha=1.2,1.8$ are depicted in Figures 1 and 2, where each frame represents a snapshot at times $t=0.5,1$ and 2 respectively. The L^1-error of the numerical solutions for $x=0$, compared to numerical estimates based on (4.1) and standard numerical estimates of the stable density, decays linearly with h. For $h=0.01$ the L^1-error is less than 0.05 for $\alpha=1.2$, and less than 0.004 for $\alpha=1.8$, for every case plotted. A short Matlab code to compute the numerical solution is included in the appendix.

Remark 5.1. It is interesting to note that the matrix in (5.4) is essentially the rate matrix of a discrete state Markov process in continuous time. Extending the state space to $N=\infty$, we obtain a Markov process Z_t^h on the state space $\{ih : i>0\}$ that approximates the reflected stable process, with $u_h(x_i,t)=\mathbb{P}(Z_t^h=ih)$. The transition rate from state ih to state jh for $i>j>1$ is $w_{i-j+1}^{\alpha}\approx \alpha(\alpha-1)(i-j)^{-\alpha-1}/\Gamma(2-\alpha)$, the jump intensity of the stable process Y_t, in view of [26, Eq. (2.5)]. The transition rate from state ih for $i>1$ to state jh for $j=1$, in the first row of the rate matrix, is $w_{i-1}^{\alpha-1}\approx i^{-\alpha}/\Gamma(1-\alpha)$, the rate at which the process Y_t would jump into the negative half-line. This can be computed as $\phi(-\infty,-ih)$ where $\phi(dy)=\alpha(\alpha-1)|y|^{-1-\alpha}dy/\Gamma(2-\alpha)$ is the Lévy measure of the process Y_t, e.g., see [26, Proposition 3.12].

Remark 5.2. The fractional boundary condition in (5.1) is a natural extension of the boundary condition for Brownian motion on the half-line. Let $u(y,t)$ denote the transition density of reflected Brownian motion, the process Z_t when $\alpha=2$. Then

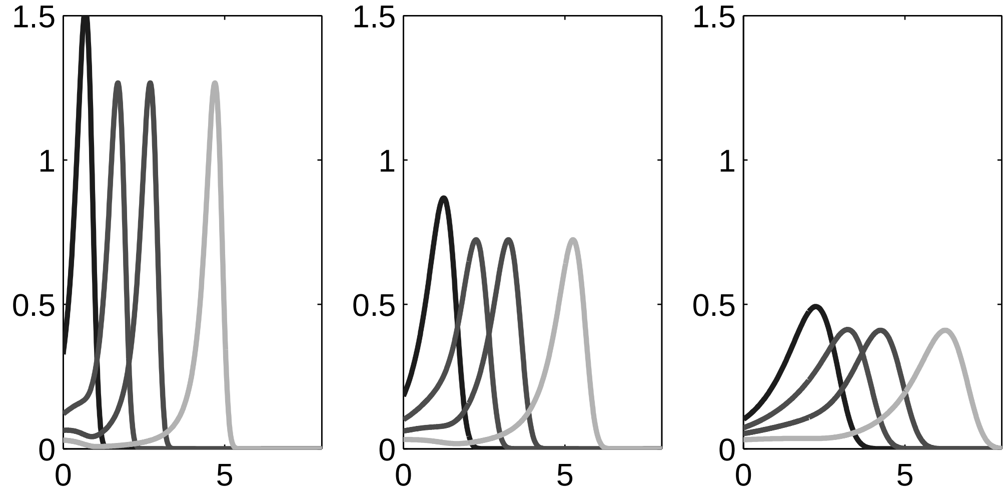

FIGURE 1. The transition densities of $p(x, y, t)$ with $\alpha = 1.2$, $x = 0, 1, 2, 4$ (left to right) at times $t = 0.5$ (left panel), $t = 1$ (middle panel), and $t = 2$ (right panel).

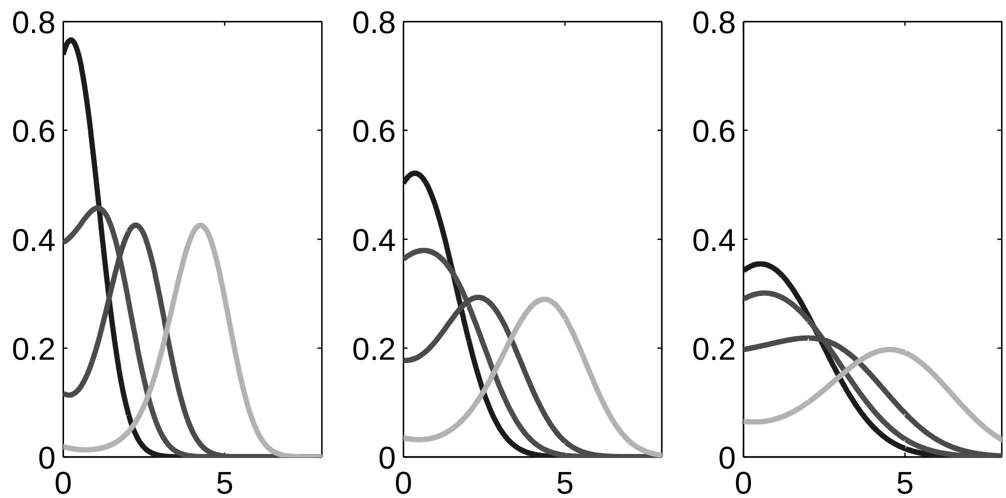

FIGURE 2. The transition densities of $p(x, y, t)$ with $\alpha = 1.8$, $x = 0, 1, 2, 4$ (left to right) at times $t = 0.5$ (left panel), $t = 1$ (middle panel), and $t = 2$ (right panel).

$\partial_t u(y, t) = \partial_y^2 u(y, t)$ and

$$\lim_{h \to 0+} \frac{u(y+h, t) - u(y, t)}{h} = 0 \quad \text{at } x = 0, \text{ for all } t > 0, \tag{5.5}$$

see for example Itô and McKean [18, Eq. 8]. The fractional boundary condition in (5.1) can be written in Grünwald finite difference form [26, Proposition 2.1]

$$\lim_{h\to 0+} \frac{1}{h^{\alpha-1}} \sum_{k=0}^{\infty} w_k^{\alpha-1} u(y+kh,t) = 0 \quad \text{at } y=0, \text{ for all } t>0, \tag{5.6}$$

using the (fractional) binomial coefficients defined in (5.2). When $\alpha = 2$ we have $w_0^{\alpha-1} = 1$, $w_1^{\alpha-1} = -1$, and $w_k^{\alpha-1} = 0$ for $k > 1$, so that (5.6) reduces to the classical condition (5.5), i.e., the one-sided first derivative. In either case ($\alpha = 2$ or $1 < \alpha < 2$), the boundary term enforces a no-flux condition at the point $y = 0$ in the state space.

Remark 5.3. Bernyk, Dalang and Peskir [9, Appendix] computed the backward generator of a general reflected stable Lévy process. Caballero and Chaumont [11, Theorem 3] compute the backward generator of a killed stable Lévy process. It may be possible to develop the forward equation and compute the transition density for those process, using the methods of this paper. This would be interesting for applications to fractional diffusion, since it could elucidate the relevant fractional boundary conditions.

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Appendix

The following Matlab code computes the transition density $p(x, y, t)$ in the case $x = 2$ for the reflected stable process Z_t defined by (3.3), where Y_t is a stable Lévy process with characteristic function (3.2) and index $1 < \alpha < 2$. This code was used to generate the plots in Figures 1 and 2.

```
%%% Matlab script to compute p(x,y,t)
%% enter variables
    alpha=1.2; ymax=12; N=1200; t=[0,.5,1,2]; x=2;
%% initialise parameters
    h=ymax/N; y=(h:h:ymax)';
    u0=zeros(N,1);u0(floor(x/h)+1)=1/h; % initial condition
%% Make Grunwald matrix
    w=ones(1,N+1);
    for k=1:N
      w(k+1)=w(k)*(k-alpha-1)/k;
    end
    w=w/h^alpha;
    M=spdiags(repmat(w,N,1),-1:1:N-1,N,N); %enter w's along diagonals
    M(1,:)=-cumsum(w(1:N))'; %change first row for BC
%% Solve ODE system
    [~,p]=ode113(@(t,u) M*u,t,u0);
```

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